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The Christoffel function for the Hermite weight is bell-shaped ☆

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Abstract

We show that the Christoffel function λ_n associated with the Hermite weight function $w_H(x) = \exp(-x^2)$ is bell-shaped. As a consequence, we describe completely how the weights in a Gauss-type quadrature formula associated with $w_H(x)$ are arranged in magnitude.
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1. The result

Let $d\alpha$ be a finite positive Borel measure on the real line, such that all its moments are finite, i.e.,

$$\mu_n = \int_{\mathbb{R}} x^n \, d\alpha(x) < \infty, \quad n = 0, 1, 2, \ldots, \ (\mu_0 > 0).$$

Denote by $\{p_n\}_{n=0}^{\infty}$ the associated sequence of polynomials orthonormal with respect to $d\alpha$ on $\mathbb{R}$, i.e., polynomials

$$p_n(x) = p_n(d\alpha; x) = \gamma_n x^n + \ldots, \quad \gamma_n = \gamma_n(d\alpha) > 0,$$

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satisfying

$$\int_{\mathbb{R}} p_m(x)p_n(x)\,d\alpha(x) = \delta_{mn}, \quad m,n = 0,1,2,\ldots\ .$$

The Christoffel function associated with $d\alpha$ is defined by

$$\lambda_n(d\alpha;x) := \left[\sum_{k=0}^{n-1} p_k(d\alpha;x)^2\right]^{-1}, \quad n = 1,2,\ldots,$$

or, equivalently, by

$$\lambda_n(d\alpha;x) = \frac{\gamma_n}{\gamma_{n-1}} \frac{1}{p_n'(d\alpha;x)p_{n-1}(d\alpha;x) - p_n(d\alpha;x)p_{n-1}'(d\alpha;x)}.$$

For a detailed account on Christoffel functions and their application to various problems arising in orthogonal polynomials, approximation theory, harmonic and numerical analysis, we refer the reader to the survey of Nevai [6].

This note is concerned with the behavior of $\lambda_n(x)$ in the case of Hermite weight function $w_H(x) = \exp(-x^2)$, i.e., $d\alpha(x) = w_H(x)\,dx$. As is well-known (see, e.g., [8]), in this case the associated orthogonal polynomials are the Hermite polynomials $\{H_n\}$ given by

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}\{e^{-x^2}\}, \quad k = 0,1,\ldots,$$

and the Christoffel function $\lambda_n(x) = \lambda_n(w_H;x)$ is

$$\lambda_n(w_H;x) = \frac{\pi^{1/2}2^n(n-1)!}{H_n'(x)H_{n-1}(x) - H_n(x)H_{n-1}'(x)}. \tag{1.1}$$

It turns out that the graph of $\lambda_n(w_H;x)$ is extremely simple, a fact that seems not to have been noticed before.

Theorem 1. *The Christoffel function $\lambda_n(w_H;x)$ is bell-shaped.*

Proof. All we need is to show that $\lambda_n'(w_H;x)$ changes its sign only at $x = 0$. The proof makes use of the following "individual" properties of the Hermite polynomials (cf. [8, Chapter 5.5])

$$H_k'(x) = 2kH_{k-1}(x) \tag{1.2}$$

and

$$H_k''(x) = 2xH_k'(x) - 2kH_k(x). \tag{1.3}$$

Clearly, $\lambda_n'(w_H;x) = 0$ if and only if $H_n''(x)H_{n-1}(x) - H_n(x)H_{n-1}''(x) = 0$. With the help of (1.2) and (1.3) we find

$$H_n''H_{n-1} - H_nH_{n-1}'' = -\frac{H_n^2}{2n}\left(\frac{H_n''}{H_n}\right)',$$

$$\left(\frac{H_n''}{H_n}\right)' = \left(2x\frac{H_n'}{H_n} - 2n\right)' = 2\left[\frac{H_n'}{H_n} + x\left(\frac{H_n'}{H_n}\right)'\right],$$

hence

$$H_n''H_{n-1} - H_nH_{n-1}'' = -\frac{H_n^2}{n}\left[\frac{H_n'}{H_n} + x\left(\frac{H_n'}{H_n}\right)'\right].$$

With $x_{1,n} < \cdots < x_{n,n}$ being the zeros of H_n (by symmetry, $x_{n+1-i,n} = -x_{i,n}$ for $i = 1, \dots, n$), we obtain

$$\begin{aligned}
\frac{H_n'}{H_n} + x\left(\frac{H_n'}{H_n}\right)' &= \sum_{i=1}^{n} \frac{1}{x - x_{i,n}} - \sum_{i=1}^{n} \frac{x}{(x - x_{i,n})^2} \\
&= -\sum_{i=1}^{n} \frac{x_{i,n}}{(x - x_{i,n})^2} \\
&= -\frac{1}{2}\sum_{i=1}^{n}\left[\frac{x_{i,n}}{(x - x_{i,n})^2} - \frac{x_{i,n}}{(x + x_{i,n})^2}\right] \\
&= -2x\sum_{i=1}^{n} \frac{x_{i,n}^2}{(x^2 - x_{i,n}^2)^2}.
\end{aligned}$$

Hence, we found

$$H_n''(x)H_{n-1}(x) - H_n(x)H_{n-1}''(x) = \frac{2xH_n^2(x)}{n}\sum_{i=1}^{n}\left(\frac{x_{i,n}}{x^2 - x_{i,n}^2}\right)^2,$$

showing that $\lambda_n'(w_H; x)$ changes its sign only at $x = 0$, and $x = 0$ is a single or triple zero of $\lambda_n'(w_H; x)$ depending on whether n is even or odd number. Theorem 1 is proved. □

2. Application to quadrature formulae

The Christoffel function $\lambda_n(d\alpha)$ is closely related to the weights $\lambda_{k,n} = \lambda_{k,n}(d\alpha)$ $(k = 1, \dots, n)$ of the n-point Gauss-Jacobi quadrature formula

$$Q_n^{\mathrm{G}}[f] = \sum_{k=1}^{n} \lambda_{k,n} f(x_{k,n}), \tag{2.1}$$

which is determined uniquely by the property that it calculates exactly the integral

$$I[f] = I[d\alpha; f] := \int_{\mathbb{R}} f(x)\,d\alpha(x)$$

whenever f is algebraic polynomial of degree not exceeding $2n - 1$, and wherein $\{x_{k,n}\}_{k=1}^{n} = \{x_{k,n}(d\alpha)\}_{k=1}^{n}$ are the zeros of $p_n(d\alpha)$, $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$. Namely, the weights $\lambda_{k,n}$ in Q_n^{G} are given by

$$\lambda_{k,n}(d\alpha) = \lambda_n(d\alpha; x_{k,n}(d\alpha)), \quad k = 1, \dots, n.$$

This is a particular case of a more general property of the Christoffel function $\lambda_n(d\alpha)$, which we recall below.

Definition 2. A quadrature formula

$$Q_n[f] = \sum_{k=1}^{n} a_{k,n} f(t_{k,n}), \quad t_{1,n} < t_{2,n} < \cdots < t_{n,n} \tag{2.2}$$

is said to be a *Gauss-type quadrature formula associated with* $d\alpha$, if $Q_n[f] = I[f]$ for every polynomial f of degree not exceeding $2n-2$.

As is well-known, a necessary and sufficient condition for (2.2) to be of Gauss-type is (2.2) to be interpolatory with nodes generated by the zeros of a polynomial

$$\psi_n(\varrho; x) = p_n(d\alpha; x) - \varrho p_{n-1}(d\alpha; x), \quad \varrho \in \mathbb{R}. \tag{2.3}$$

For each $\varrho \in \mathbb{R}$ the zeros $\{t_{k,n}(\varrho)\}_{k=1}^{n}$ of ψ_n are real, distinct and $\{t_{k,n}(\varrho)\}$ are strictly monotone increasing functions. This follows easily from the zero separation property of orthogonal polynomials. As a consequence of this monotonicity, for $k = 1, \dots, n$ the kth node of a Gauss-type quadrature formula belongs to $(x_{k-1,n-1}(d\alpha), x_{k,n-1}(d\alpha))$, where $x_{0,n-1}(d\alpha) := -\infty$ and $x_{n,n-1}(d\alpha) := \infty$. Another consequence is that the nodes of every two Gauss-type quadrature formulae interlace.

We shall use the following important property of the Christoffel function (see, e.g., [4, Chapter 1]):

Proposition 3. *Given an arbitrary* $\xi \in \mathbb{R}$ *such that* $p_{n-1}(d\alpha; \xi) \neq 0$, *there exists a unique Gauss-type quadrature formula* (2.2) *which has* ξ *as a node. The corresponding to* ξ *weight in this quadrature formula is equal to* $\lambda_n(d\alpha; \xi)$.

In view of Theorem 1, for the weights in the Gauss-type quadrature formulae associated with the Hermite weight function $w_H(x) = \exp(-x^2)$ we have the following *comparison rule*: *the closer the node to the origin, the larger the corresponding weight*. Winston [9] has proved this property for the weights of the Gauss–Jacobi quadrature formulae associated with $w_H(x)$, using an approach proposed by Sonin [7]. Since $\lambda_{k,n-1} = \lambda_{n-1}(x_{k,n-1}) = \lambda_n(x_{k,n-1})$ for $k = 1, \dots, n-1$, we obtain Winston's result as a consequence of our comparison rule:

Corollary 4. *The weights of the Gaussian quadrature formulae* Q_n^{G} *and* Q_{n-1}^{G} *associated with the Hermite weight function* $w_H(x) = \exp(-x^2)$ *satisfy*

$$\lambda_{1,n} < \lambda_{1,n-1} < \lambda_{2,n} < \lambda_{2,n-1} \cdots < \lambda_{[(m+1)/2],m},$$

where $m = n-1$ *if* n *is even, and* $m = n$ *if* n *is odd.*

For nonsymmetric Gauss-type quadrature formulae our comparison rule yields:

Corollary 5. *Let* (2.2) *be a non-symmetric Gauss-type quadrature formula associated with the Hermite weight function* $w_H(x) = \exp(-x^2)$.

(a) *If for a* $k \in \{1, 2, \dots, [n/2]\}$ $t_{n+1-k,n} > -t_{k,n}$, *then*

$$a_{n,n} < a_{1,n} < a_{n-1,n} < a_{2,n} < a_{n-2,n} < \cdots.$$

(b) *If for a* $k \in \{1, 2, \dots, [n/2]\}$ $t_{n+1-k,n} < -t_{k,n}$, *then*

$$a_{1,n} < a_{n,n} < a_{2,n} < a_{n-1,n} < a_{3,n} < \cdots.$$

Proof. Actually, we prove a slightly stronger result. Consider case (a). Since the nodes of Q_n and Q_n^{G} interlace, the inequality $t_{n+1-k,n} > -t_{k,n}$ shows that the nodes of Q_n are biased to the right with respect to the nodes of Q_n^{G}. We make use of the symmetrical structure of Q_n^{G} and Q_{n-1}^{G}, the fact that the nodes of Q_n interlace with the nodes of both Q_n^{G} and Q_{n-1}^{G}, and the comparison rule to obtain the inequalities

$$a_{n,n} < \lambda_{n,n} = \lambda_{1,n} < a_{1,n} < \lambda_{1,n-1} = \lambda_{n-1,n-1} < a_{n-1,n} < \lambda_{n-1,n},$$

$$\lambda_{n-1,n} = \lambda_{2,n} < a_{2,n} < \lambda_{2,n-1} = \lambda_{n-2,n-1} < a_{n-2,n} < \lambda_{n-2,n}.$$

Proceeding in the same manner, we find the arrangement of all the weights of Q_n. The proof of part (b) is analogous, and therefore is omitted. □

Many properties of the Christoffel function associated with the ultraspherical weight function $w_\mu(x) = (1-x^2)^{\mu-1/2}$, including various inequalities between the weights of the related Gauss, Radau and Lobatto quadrature formulae, have been obtained by Förster [2]. Unlike the situation with Theorem 1, the graph of the Christoffel function $\lambda_n(w_\mu; x)$ is more complicated, e.g., for $-1/2 < \mu \leqslant 1/2$, $\lambda_n(w_\mu; x)$ has exactly $2n-3$ local extrema (see [2]). Nevertheless, it is well-known (cf., e.g., [8, Chapter 15]) that the weights of the Gauss–Jacobi quadrature formula Q_n^{G} associated with $w_\mu (\mu > 0)$ increase as their abscisae approach the origin, and have the opposite behavior when $\mu < 0$. This fact has been used in [1] in the study of the weight distribution of spherical t-designs.

For other inequalities and estimates involving the weights in Gauss-type quadrature formulae associated with the classical weight functions of Jacobi, Laguerre and Hermite we refer the reader to [3,5,9]. Note that all these papers make use of the Sonin approach, which is not applicable if the associated orthogonal polynomials do not satisfy a second-order differential equation. We conjecture that the bell-shaped form in the Christoffel function persists for more general weight functions, e.g., for Freud weights of the form $w(x) = \exp(x^{-m})$, m—even positive integer. However, for $m > 2$ the proof must rely on different kind of arguments.

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